LEFM

Stress (Cartesian) : $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$
Strain (") : $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$

Pl. σ : $\sigma_z = \tau_{xy} = \tau_{yz} = 0$, $E' = E$, $\nu' = \nu$
Pl ε : $\varepsilon_z = 0$, $E' = \dfrac{E}{1-\nu^2}$, $\nu' = \dfrac{\nu}{1-\nu}$

<u>Equilibrium</u>

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \qquad (1)$$

where X, Y = body force

<u>Strain - Disp.</u>

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{where } u, v = \text{disp. comp.} \quad (2)$$

<u>Stress - Strain</u>

$$\varepsilon_x = \frac{1}{E'}(\sigma_x - \nu' \sigma_y) \qquad \varepsilon_y = \frac{1}{E'}(\sigma_y - \nu' \sigma_x) \qquad \gamma_{xy} = \frac{2(1+\nu')}{E'} \tau_{xy} \qquad (3)$$

If X, Y (body force) = constant, Satisfying BC (Automatically)

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2} - Xx \qquad \sigma_y = \frac{\partial^2 \psi}{\partial x^2} - Yy \qquad \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y} \qquad (4)$$

where ψ = Airy Stress Function.

Substitute (2) + (4) → (3) + Differentiate Twice

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0. \qquad \Leftarrow \text{Governing Equation}$$

⇒ Find an Airy Stress Function that satisfies the Governing Equation.

Westergaard Solution (Based on Muskhelishvili)

Complex Function $Z(z) = Z_r(x,y) + i Z_i(x,y)$

where $z = x + iy$, r = real, i = imaginary.

Cauchy - Riemann Cond.

$$\frac{\partial Z_r(x,y)}{\partial x} = \frac{\partial Z_i(x,y)}{\partial y} \qquad \frac{\partial Z_i(x,y)}{\partial x} = -\frac{\partial Z_r(x,y)}{\partial y}$$

Transform Governing Eq. as function of z

$$\frac{\partial^4 \psi}{\partial z^2 \partial \bar{z}^2} = 0. \qquad \text{where} \quad \begin{array}{l} z = x + iy \\ \bar{z} = x - iy. \end{array}$$

Use. $$\psi = \frac{1}{2}[\bar{z}\varphi_1(z) + z\overline{\varphi_1(z)} + \varphi_2(z) + \overline{\varphi_2(z)}] \qquad (5)$$

where $\varphi_1(z), \varphi_2(z)$ = analytic functions satisfying Cauchy - Riemann cond.

Substitute (5) → (4) & neglect body force

$$\sigma_x + \sigma_y = 2[\varphi_1'(z) + \overline{\varphi_1'(z)}]$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\varphi_1''(z) + \varphi_3'(z)]$$

where prime $(') = \frac{\partial}{\partial z}$, $\varphi_3(z) = \frac{\partial \varphi_2(z)}{\partial z}$

Stress Component ⇒ Separate Real & Imaginary Part

Disp. " ⇒ " " " "